CHAPTER 2.6

The origins of action-angle variables and Bohr's introduction of them in a 1918 paper

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Abstract

Action-angle variables are strong mathematical tools for discussing the old quantum theory. In a 1918 memoir, Bohr used the action variables and explained in a footnote their importance in the quantum context. The idea of the action-angle variables appeared in celestial mechanics at the beginning of the twentieth century and grew within the old quantum theory thanks to the efforts of Schwarzschild and Epstein on the Stark problem. Its development was closely connected to discoveries of the mathematical properties of systems for which the Hamilton-Jacobi equation is completely separable. Our investigation also clarifies how notions that appeared in Bohr's memoir were established, concerning in particular the relationship between properties of the Hamilton-Jacobi equation and conditionally periodic motion, the treatment of degenerate systems, and action-angle variables.

Key words: Stark effect; conditionally periodic motion; Karl Schwarzschild; Paul Epstein; Carl V. L. Charlier; Hamilton-Jacobi equation; Stäckel's formulas.

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1. Introduction

In 1918, Niels Bohr published an important memoir in three parts titled "On the Quantum Theory of Line-Spectra". The first part of the paper was devoted to periodic systems with one degree of freedom and conditionally periodic motions. Bohr knew conditionally periodic motions occur if the system is described by a Hamilton-Jacobi equation for which the variables were completely separable. Bohr assumed such a system and, on introducing the so-called action variables that have the same dimension as action, explained properties of the system. While noting the relation between the action variables and invariability for slow mechanical transformations, Bohr mentioned the independent work of Schwarzschild and Epstein on the Stark effect for which the action integral were related to the so-called angle variables.[1] In a footnote, Bohr explained the canonical variables, which are composed of action variables and angle variables.[2] His footnote covers the essentials of the modern definition of action-angle variables.[3] Bohr's 1918 paper paid special attention, not to these canonical variables, but to action variables. However, he regarded action variables as being constructed from canonical variables with angle variables. This paper examines the historical development in constructing the action-angle variables.

Angle variables had been used in celestial mechanics independently of their use in the Hamilton-Jacobi theory.[4] Our interest is in the process through which angle variables when paired with action variables came to be treated as canonical variables. To begin, we shall recall this process as it evolved in celestial mechanics. In developing the early Hamilton-Jacobi theory, mathematicians noted a

1. Bohr (1918), pp. 21-22.

2. Bohr (1918), pp. 29-30.

3. For example, Goldstein, Poole, and Safko (2002), pp. 452-463.

4. We confirm that the action variables were used in celestial mechanics at latest at the end of the nineteenth century in Whittaker (1899), pp.153-154. However, these variables were not treated as canonical variables whose counterparts were action variables. Jammer (1966), p. 103, seemed to confuse these facts and wrote that action-angle variables were fully recognized in astronomy in the era of Poincaré and Charlier.

system for which the variables of the associated Hamilton-Jacobi equation were completely separable, and obtained a couple of mathematical properties guaranteeing that conditionally periodic motion occurs in the system.

In parallel, the Hamilton-Jacobi theory at the end of the nineteenth century began to involve aspects of transformation theory. Rearranging the results of his predecessors, such as Jacobi, Paul Stäckel, Hugo Gyldén, and Henri Poincaré, Carl Vilhelm Ludwig Charlier derived, in the second volume of his textbook series on celestial mechanics published in 1907, a special set of the canonical variables composed of the action integral and the angle variable. These variables were generalized and introduced into quantum theory by Karl Schwarzschild in a 1916 paper that explained the Stark effect. Paul Epstein published his results on the Stark effect almost at the same time. After encountering Schwarzschild's ideas, Epstein rearranged and expanded on them. Next, we shall examine the work of these two scientists who Bohr mentioned in his introduction of his 1918 paper.

There are several historical articles that discuss Schwarzschild's usage of action-angle variables in the Stark problem. These articles explain his rather modern introduction of the action-angle variables.[5] However, Schwarzschild did not hold this modern understanding of these variables when he analyzed the Stark problem. These canonical variables were actually formulated by two others working on the same problem. In truth, Hamiltonian dynamics in general, and the Hamilton-Jacobi equation in particular, was broadly used in the texts on celestial mechanics and astronomy.[6] Nevertheless, action-angle variables were also required in considering the old quantum mechanics.

Our examination also clarifies how ideas and notions that appear in the 1918 paper emerged in constructing the action-angle variables. We specifically note the relationship between properties

5. For example, Duncan and Janssen (2014a), Eckert (2013a), pp. 44-48, (2013b), pp. 210-213, Darrigol (1992), pp. 113-116, Mehra and Rechenberg (1982), pp. 223-227, Hund (1974) ,pp. 85-86, and Jammer (1966), pp. 102-104.

6. Shore (2003), p. 498. Shore's paper discusses the aspect of how the classical dynamical problem linked to the old quantum theory. He also notes the Stark problem but pays little attention to action-angle variables.

of the Hamilton-Jacobi equation and conditionally periodic motion, and the relationship among the notions of degenerate systems, angle variables, and quantum numbers. The idea for the adiabatic hypothesis, which Bohr also mentioned in his 1918 paper, can be related to the history of action-angle variables; however, we shall not pursue this point here.

2. The origin of action-angle variables: celestial mechanics

2.1 Jacobi's achievements

Jacobi elaborated on Hamilton's works on mechanics[7] published in 1834 and 1835. Jacobi rearranged his results and then delivered lectures on dynamics in Königsberg, titled *Vorlesungen über Dynamik* in 1842-1843. These lectures provided a prototype of the Hamilton-Jacobi theory. Among Jacobi's accomplishments, we focus on subjects related to the solutions of the Hamilton-Jacobi equation.

It should first be noted that Jacobi clarified a fundamental idea of the Hamilton-Jacobi theory. The idea was a way to reduce the solutions of a system of ordinary differential equations, the canonical equations,[8]

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad i = 1, 2, \ldots, n, \quad H = H(q_1, \ldots, q_n, p_1, \ldots, p_n), \tag{1}$$

to those of a special type of first-order, non-linear, partial differential equation, the Hamilton-Jacobi equation,

$$H\left(q_1, \ldots, q_n, \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_n}\right) = \gamma_1, \qquad p_i = \frac{\partial W}{\partial q_i}. \tag{2}$$

The complete integral has the form $W = W(q_1, \ldots, q_n, \gamma_1, \ldots, \gamma_n)$, where $\gamma_1, \ldots, \gamma_n$ are arbitrary constants and γ_1 can be taken to be the total energy. The solutions of Eq. 1 are given by

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7. For this process, see Nakane and Fraser (2002).

8. Jacobi (1866) discussed initially the case when the function explicitly depends on time and then formulated the time-independent case. As the function H is time-independent in dynamical problems, we are only concerned with the latter.

$$H\left(q_1,\ldots,q_n,\frac{\partial W}{\partial q_1},\ldots,\frac{\partial W}{\partial q_n}\right)=\gamma_1, \qquad p_i=\frac{\partial W}{\partial q_i}.$$

where the β_1 s are arbitrary constants. In demonstrating these relations, Jacobi indicated that W corresponded to the action integral of the system in the dynamical theory.[9]

In general, solving a partial differential equation is more difficult than solving a system of ordinary differential equations, but Jacobi noted that if one finds appropriate coordinates that completely separate the variables of the Hamilton-Jacobi equation, one can easily solve it. Jacobi showed that by using polar coordinates the Hamilton-Jacobi equation for the Kepler problem was completely separable. Jacobi appealed to elliptical coordinates to solve the equations of motion of a particle gravitationally attracted by two fixed centers, a problem proposed by Leonhard Euler.[10] It was one of the most remarkable achievements of the Hamilton-Jacobi theory. However, Jacobi realized that his method was very limited. He confessed that there was no general way to find such coordinates for a given Hamilton-Jacobi equation.

Instances when the variables are completely separable are very important for our argument. We often describe the Hamilton-Jacobi equation as solvable if its variables are completely separable. We shall discuss two cases related to solvable Hamilton-Jacobi equations: one in which mathematicians and physicists succeeded in finding appropriate coordinates to solve the equation; the other in which they assumed that a given Hamilton-Jacobi equation has such coordinates and developed their discussions.

Jacobi's discussion of the Hamilton-Jacobi equations is reminiscent, for those who know modern Hamilton-Jacobi theory, of the idea of canonical transformations. These involve changes of variables that conserve the original form of the canonical equations. The

9. Hamilton's original idea was that solutions of equations of motion are derived from the action integral of the system W obtained by solving a special type of first-order partial differential equation. See Nakane and Fraser (2002).
10. Houzel (1978) discusses the history of this problem.

complete integral of the associated Hamilton-Jacobi equation involves such a transformation. In his 1837 paper, Jacobi characterized what is now called the generating function of the canonical transformation.[11] However, in all his papers referred to in the early 20th century, Jacobi never indicated that a complete integral of the Hamilton-Jacobi equation played the role of a generating function[12] Not only mathematicians and physicists but also historians often miss this fact.[13]

2.2 Stäckel's and Poincaré's results

Jacobi's ideas blossomed in the late nineteenth century. One most important event for the quantum theory was Stäckel's publication in 1893 of several mathematical properties of a separable Hamilton-Jacobi equation. His main result, which Charlier called Stäckel's theorem, is as follows:[14] a condition for the separability of variables in

$$\sum_{i=1}^{n} A_i \left(\frac{\partial W}{\partial q_i} \right)^2 - 2(U - \gamma_1) = 0$$

is the existence of arbitrary functions of one variable

$$\varphi_{ij}(q_i), \quad \psi_i(q_i), \quad i, j = 1, 2, \ldots, n,$$

where $\Delta = |\varphi_{ij}(q_j)|$ does not identically vanish and the functions, $A_1, \ldots, A_n$, and U are determined by

11. Hagihara (1970), pp. 53-54 indicates that the term generating function was introduced by Carathéodory in 1935. We adopt this term for the function that determines the canonical transformation, while noting that Carathéodory's definition of the canonical transformation was different from that of Jacobi.

12. Jacobi (1890), a posthumously published paper, involved a proof of his statement.

13. For example, see Kline (1972), pp. 743-744 and Mehra and Rechenberg (1982) p. 226, footnote 359. Because Jacobi's accomplishment related to the canonical transformation is overestimated, the historical discussion of the Hamilton-Jacobi theory has been confusing. Exceptionally Klein (1926), pp. 203-205, grasped precisely Jacobi's accomplishment and did not connect Jacobi's transformation with a complete solution of the Hamilton-Jacobi equation.

14. Charlier (1902), pp. 80-81. The present paper adopts Stäckel's results, which were introduced in Charlier's book of 1902.

$$A_i = \frac{1}{\Delta}\frac{\partial\Delta}{\partial\varphi_{i1}} \quad (i = 1, 2, \ldots, n), \quad U = \sum_{i=1}^{n} \psi_i \frac{1}{\Delta}\frac{\partial\Delta}{\partial\varphi_{i1}}.$$

In this case,

$$W = \sum_{i=1}^{n} \int \sqrt{2\psi_i(q_i) + \sum_{\lambda=1}^{n} 2\gamma_\lambda \varphi_{i\lambda}(q_i)}\, dq_i, \tag{3}$$

where the γ_λ s are integral constants, gives a complete solution of Eq. 2.

Stäckel thus provided a formula for solving the Hamilton-Jacobi equations. Solving a given Hamilton-Jacobi equation using this formula however is not practical, although it was effective for examining the general properties of systems associated with solvable Hamilton-Jacobi equations. Stäckel demonstrated that such a system was composed of multiple one-dimensional periodic motions. In accordance with the terminology of Otto Staude, Stäckel called these motions conditionally periodic motions.[15] As we shall see below, the relation between a solvable Hamilton-Jacobi equation and conditionally periodic motion is a key relationship that the theory would be noted for in the old quantum theory.

Jacobi's idea of canonical transformations was also developed further in celestial mechanics. In *les Méthodes Nouvelles de la Mécanique Céleste* published in 1892, Poincaré first introduced Jacobi's first theorem that showed a way to solve Eq. 1 using a complete solution of the associated Hamilton-Jacobi Eq. 2. He next indicated that the complete solution becomes a generating function of the canonical transformation[16] which transforms Eq. 1 to

15. Stäckel (1893), p.554. Let us consider a system of two one-dimensional motions for which the frequencies are ω_1 and ω_2. If ω_1 / ω_2 is irrational the orbit returns infinitely close to the starting point after a sufficiently long duration. The motion is not periodic but very similar to being periodic. Additionally, a periodic motion occurs if ω_1 / ω_2 is rational; the system involving such a motion is called conditionally periodic. See Arnold (1989), pp. 285-287

16. Poincaré did not obtain the correct relationship among the variables of the generating function, the old canonical variables, and new ones. Although his description was confusing, I have adopted the relations that he introduced when deriving Delaunay's equations as demonstrated in Poincaré (1892),pp. 24-26.

$$\frac{d\gamma_i}{dt}=\frac{\partial H}{\partial \beta_i},\frac{d\beta_i}{dt}=-\frac{\partial H}{\partial \gamma_i},\quad i=1,2,\ldots,n,\quad H=H(\gamma_1,\ldots,\gamma_n,\beta_1,\ldots,\beta_n).$$

Poincaré called it Jacobi's second theorem even though Jacobi did not derive it. It is Poincaré who noted that the complete solution works as a generating function. Poincaré actually performed the canonical transformation and derived the other type of canonical equation, Delaunay's equation,[17] from Eq. 1, using Jacobi's second theorem. However, Poincaré did not prove Jacobi's second theorem in any of his books and papers.

2.3 Charlier's textbooks on celestial mechanics

Jacobi's theory was mainly developed in mathematics and celestial mechanics. Quantum physicists knew the latest Hamilton-Jacobi theory through Charlier's work. Charlier introduced Stäckel's and Poincaré's achievements in his book *Die Mechanik des Himmels*. At the end of the nineteenth century, several books on celestial mechanics were published.[18] Among them, Charlier's books concentrated on discussing properties of dynamic systems associated with solvable Hamilton-Jacobi equations. He also demonstrated several ways to reduce any system to a system associated with solvable Hamilton-Jacobi equations. For example, he effectively used Gyldén's idea of intermediate orbit, which is an approximate orbit to the original and is associated with a solvable Hamilton-Jacobi equation.[19]

17. Delaunay's equations are

$$\begin{cases}\dfrac{dL}{dt}=-\dfrac{\partial H}{\partial l}, & \dfrac{dG}{dt}=-\dfrac{\partial H}{\partial g}, & \dfrac{d\Theta}{dt}=-\dfrac{\partial H}{\partial \theta},\\[2ex] \dfrac{dl}{dt}=\dfrac{\partial H}{\partial L}, & \dfrac{dg}{dt}=\dfrac{\partial H}{\partial G}, & \dfrac{d\theta}{dt}=\dfrac{\partial H}{\partial \Theta}\end{cases}$$

where θ is the longitude of the node, $g+\theta$ that of the perihelion, l the mean anomaly, and $L=\sqrt{a}, G=\sqrt{a(1-e^2)}, \Theta=G\cos i$, with a, e, and i denote the major axis, the eccentricity, and the inclination. See Poincaré (1892), p. 25.

18. See Shore (2003).

19. Gyldén (1841-1896) was a Finnish-Swedish astronomer, a leading theorist of celestial mechanics and planetary perturbations. See Markkanen (2009). Whittaker (1899) pp. 138-144 and Poincaré (1893), vol.2, pp. 202-227 devoted a chapter to

The last chapter of the second volume of Charlier's book began with transformation theory. In accordance with Poincaré's suggestion, Charlier proved that a complete solution of the Hamilton-Jacobi equation leads to a canonical transformation, Jacobi's second theorem. However, this theorem does not ensure that one can actually solve the given Hamilton-Jacobi equation. Charlier then proved that a complete solution of Gyldén's intermediate orbit can also become a generating function of the canonical transformation. After the transformation, he obtained new canonical variables. Noting Stäckel's relation between a solvable Hamilton-Jacobi equation and conditionally periodic motion, he modified the new variables. Finally, he derived the canonical variables that were composed of a linear function of time and an element of W in Eq. 3, an action integral of motion, divided by π, specifically,

$$\eta_i = -\frac{\partial C}{\partial \xi_i} t + c_i, \quad \xi_i = \frac{1}{\pi}\int_{a_i}^{b_i} \sqrt{2\psi_i(q_i) + \sum_{j=1}^{n} 2\gamma_j \phi_{ij}(q_i)}\, dq_i, \quad i = 1, \ldots, n$$

where C is the total energy that appears in the equation of the intermediate orbit, c_is are arbitrary constants, and q_i oscillates between a_i and b_i.

Charlier applied his transformation theory to the motion in three-dimensional space. He chose an elliptical orbit as the intermediate orbit and performed the canonical transformation. The coefficient of time then becomes the so-called mean motion.[20] The new canonical coordinates were composed of angle variables and action integrals divided by π. This is the origin of the action-angle variables. Angle variables were very common in celestial mechanics. Charlier showed that these variables could be related to the Hamilton-Jacobi theory. Indeed, Charlier never paid any special attention to them, leaving it to Schwarzschild to discover their importance.

introduce Gyldén's work including his idea on absolute and intermediate orbits.

20. The mean motion n is given by $n = \frac{2\pi}{P}$ where P is the period of elliptic motion.

3. Action-angle variables and quantum theory

Now, let us examine the contributions resulting from the Stark effect published at almost the same time in 1916. The work of the two contributors related to the Stark problem has already been described[21]. Here, we confirm a well-known relation involving Sommerfeld. The German physicist was interested in the Stark effect and had discussed the topic with Schwarzschild. Sommerfeld had turned the problem of the Stark effect over to Epstein, who was staying at Sommerfeld's institute in Munich. Epstein knew of Schwarzschild's idea through Sommerfeld.

Schwarzschild and Epstein both considered the motion of an electron attracted by a fixed center of force from an atomic nucleus in a homogeneous electric field. Both of them introduced the Hamilton-Jacobi equation for the total energy as

$$H = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{\chi e^2}{r} - eEx\,, \qquad r^2 = x^2 + y^2 + z^2, \tag{4}$$

where (x, y, z) is the position of the electron of mass m, an overdot on x, y, z denotes the derivative with respect to time, $-e$ is the charge of the electron, $+\chi e$ $(\chi > 0)$ that of the nucleus, and E the intensity of the electron field.

Schwarzschild passed away on the day the paper was published. Epstein submitted a preliminary note that appeared on 15 April whereas Schwarzschild's paper appeared on 11 May.[22] Both obtained similar results involving the moving area of an electron in the Stark effect. After submitting his first full paper, "Zur Theorie des Starkeffekts" to *Annalen der Physik*, Epstein read Schwarzschild's work and published a second full paper, "Zur Quantentheorie," elaborating on Schwarzschild's results from a mathematical point of view.[23]

If one wants to know only the solutions of the Stark problem, it suffices to examine only Epstein's first paper noting that Schwarzs-

21. For example, Duncan and Janssen (2014a), Eckert (2013a), pp. 45-48, (2013b), pp. 210-213, and Mehra and Rechenberg (1982), pp. 225-227.

22. Epstein (1916a); Schwarzschild (1916).

23. Epstein (1916b); Epstein (1916c).

child had chosen the alternative use of the action-angle variables.[24] Our interest though is the formation of the action angle variables. We examine Schwarzschild's and Epstein's procedures from this viewpoint. In comparison with Epstein's first paper, his second did not give any notable argument to explain the Stark effect in the Bohr-Sommerfeld theory. This second paper involved essential arguments that bridged Schwarzschild's ideas and the modern Hamilton-Jacobi theory. Our consideration also clarifies the advantage of Schwarzschild's approach over Epstein's approach in his first paper, which Sommerfeld indicated.[25]

3.1 Schwarzschild's approach

In "Zur Quantenhypothese", Schwarzschild formulated action-angle variables, developed the notion of degeneracy using these variables, and showed that degeneracy occurs in the Stark effect. He first pointed out that the canonical system of Eq. 1 describing conditionally periodic motion could be transformed to

$$\frac{d\alpha_i}{dt} = -\frac{\partial H}{\partial \omega_i}, \quad \frac{d\omega_i}{dt} = \frac{\partial H}{\partial \alpha_i}, \quad i = 1, \ldots, n, \tag{5}$$

where α_i is an integral constant and ω_i an angle variable linear in time t and expressible as $\omega_i = n_i t + \beta_i$ with n_i a mean motion and β_i an initial value of the angle; he did not however give a way to obtain these new canonical variables. Schwarzschild apparently adopted Charlier's idea that canonical variables can be constructed from angle variables. Because $\alpha_i =$ const. with respect to time, the first equation of Eq. 5 suggests $\frac{\partial H}{\partial \omega_i} = 0$; therefore H is independent of ω_is, specifically, $H = H(\alpha_1, \ldots, \alpha_n$. In contrast, the second equation of Eq. 5 gives

$$n_i = \frac{\partial H}{\partial \alpha_i}.$$

24. Duncan and Janssen (2014b) precisely examine Epstein's procedure demonstrated in his first full paper.

25. Sommerfeld (1916), pp. 39-44; Sommerfeld (1919), pp. 500-501.

As n_i is the reciprocal of time, α_i has the same dimension as an action integral. Therefore, Schwarzschild called the set of α_i s action variables and ω_is angle variables,[26] thereby establishing these terms. He continued referring to an action as α_i and never wrote this in an action integral form in general arguments.

Schwarzschild defined a volume of phase space, an idea originating with Max Planck, in terms of these new canonical variables[27] as

$$\iint \ldots \int d\alpha_1 d\omega_1 \ldots d\alpha_n d\omega_n.$$

Then, Schwarzschild noted the case for which

$$n_i = \frac{\partial H}{\partial \alpha_i} = 0$$

holds. He called the system degenerate if it involves such a mean motion n_i. More precisely, he noted that degeneracy occurs if one can choose integers $l_1, \ldots, l_n$ (at least one of which is nonzero) for which

$$l_1 n_1 + \ldots + l_n n_n = 0 \qquad (6)$$

holds. For the degenerate system, Schwarzschild showed a way to construct new action-angle variables $(\omega_1', \ldots, \omega_n', \alpha_1', \ldots, \alpha'_n)$ from the old ones. If r $(1 \le r \le n-1)$ combinations of Eq. 6 exist for the system, one sets

$$\omega_i' = l_1^{(m)} \omega_1 + \ldots + l_n^{(m)} \omega_n, \quad m = 1, \ldots, r, \; i = 1, \ldots, r,$$

for which the mean motion is 0. The other ω_i's $(i = r+1, \ldots, n)$ are represented by appropriate linear combinations of ω_i s with integer coefficients. New action variables α_i were similarly represented as linear combinations of α_i.

Schwarzschild used these new action angle variables and consid-

26. Schwarzschild (1916), p. 549 adopted the terms *Wirkungsvariable* and *Winkelvariable*.
27. Mehra and Rechenberg (1982) , pp. 208-227 set Schwarzschild's work in the context of consideration of phase space done by Ishiwara, Wilson, and Sommerfeld.

ered the phase-space volume. He found that orbits of the motion appear dense everywhere in a low-dimensional surface in phase space whereas they can appear in all parts of phase space if the system is non-degenerate.

Here Schwarzschild showed a way to derive action-angle variables by a transformation from (q_i, p_i) to (ω_i, α_i)

$$p_i = \frac{\partial W}{\partial q_i}, \qquad \omega_i = \frac{\partial W}{\partial \alpha_i}, \qquad i = 1, \dots, n,$$

where function $W = W(q_1, \dots, q_n, \alpha_1, \dots, \alpha_n)$ leads this transformation. Schwarzschild sought the leading function that involves action variables. Poincaré and Charlier suggested that a complete solution of the Hamilton-Jacobi equation associated with Eq. 1 becomes the leading function. Although the solution involves n arbitrary constants, they do not become action variables directly. Then, Schwarzschild assumed that such auxiliary variables $\eta_1, \dots, \eta_n$ that describe the q_i s as periodic functions of $\eta_1, \dots, \eta_n$ with period 2π were introduced. Furthermore, he supposed that the leading function W has the form

$$W = \alpha_1\eta_1 + \dots + \alpha_k\eta_n + T(\alpha_1, \dots, \alpha_n, \eta_1, \dots, \eta_n), \tag{7}$$

where the α_i s are integral constants and T is a periodic function of the constants and auxiliary variables η_i with period 2π with respect to η_i. He then obtained the ω_i s and α_i s that satisfy the above-mentioned relations and which become the angle and action variables.

Schwarzschild then discussed the Stark effect. He noted that the Stark problem was a special case of Jacobi's two-center problem; that is to say, one center is taken to infinity. Because this Jacobi problem is separable in elliptical coordinates, Schwarzschild was able to introduce "elliptical coordinates for the special case":[28]

$$\lambda = \frac{r + x}{2}, \qquad \mu = \frac{r - x}{2},$$

28. Schwarzschild (1916), p. 557. There are actually parabolic coordinates, which are the limiting case of elliptic coordinates.

in the plane defined by the direction of the electric field (he chose the x-direction) and the radius vector joining the nucleus (at the origin) and the electron. He chose ϕ to describe the angle between the above-mentioned plane and a fixed plane through the x-axis. These coordinates made the Hamilton-Jacobi equation Eq. 4 completely separable. Then, for the solution W, the action integral was written in the form

$$W = W_1(\lambda) + W_2(\mu) + W_3(\phi) .$$

Schwarzschild did not describe the orbit in terms of λ, μ, and ϕ but introduced action-angle variables using W. In general, W in the form Eq. 7 is not easy to construct. However, each W_i (i = 1,2.3) is a one-variable function. He succeeded in finding the auxiliary variable η_i for each part and represented W_i as an integral of a function of η_i,[29] setting

$$W_i = \alpha_i \eta_i + T(\eta_i) , \tag{8}$$

where $T(\eta_i)$ is a periodic function of period 2π. Using these action variables, he obtained the relation

$$\frac{\partial H}{\partial \alpha_3} = \frac{1}{2}\left(\frac{\partial H}{\partial \alpha_1} + \frac{\partial H}{\partial \alpha_2} \right) \quad \text{or} \quad 2n_3 = n_1 + n_2 .$$

That is, degeneracy occurs in the Stark effect. He introduced modified action-angle variables and showed that an electron orbit appears on a two-dimensional surface in three-dimensional space.

3.2 Epstein's approach

Epstein's first full paper also succeeded in solving the Hamilton-Jacobi equation for the Stark effect. Epstein introduced parabolic coordinates

29. Schwarzschild (1916), pp. 558-561, integrated the function from $\eta_i = 0$ to η_i 2π, set α_i (i = 1,2) and obtained Eq. 8. Because $\frac{\partial W_3}{\partial \phi} = \gamma_3$, he obtained $W_3 = \gamma_3\, \phi$ and set $\alpha_3 = \gamma_3$.

$$x=\frac{\xi^2-\eta^2}{2}, \quad y=\xi\eta, \quad r=\sqrt{x^2+y^2}=\frac{\xi^2+\eta^2}{2},$$

which also made the Hamilton-Jacobi equation completely separable. His coordinates are related to those of Schwarzschild if one sets $\lambda=\xi^2, \mu=\eta^2$. Distinct from Schwarzschild, Epstein obtained the orbit of the electron in accordance with Jacobi's method through a complete solution of the equation for the problem

$$W=m\alpha\phi+\int_0^\xi \sqrt{f_1(\xi)}d\xi+\int_0^\eta \sqrt{f_2(\eta)d\eta},$$

where α is integral. He produced the same results as Schwarzschild, in that the electron moves in a lower-dimensional surface of the three-dimensional space. Next, Epstein assumed Sommerfeld's quantum condition for the problem:

$$2\sqrt{m}\int_{\xi_1}^{\xi_2} \sqrt{f_1(\xi)}d\xi=n_1h, \quad 2\sqrt{m}\int_{\eta_1}^{\eta_2} \sqrt{f_1(\eta)}d\eta=n_2h, \quad 2\sqrt{m}\int_0^{2\pi} \alpha d\varphi=n_3h,$$

where $\xi_1, \xi_2, \eta_1, \eta_2$ denote the zero points of $f(\zeta)$ and $f(\eta)$[30], h is Planck's constant, and the η_is are quantum numbers. He applied them to the Stark effect and succeeded in explaining the experimental results. In addition to Sommerfeld's application of his quantum condition to the elliptical motion, Epstein's argument suggests that the coordinates that make the associated Hamilton-Jacobi equation completely separable are appropriate coordinates for Sommerfeld's quantum conditions.

In his second paper, Epstein reformulated Schwarzschild's arguments on action-angle variables and introduced the idea of degeneracy, which Epstein never introduced in his first paper. Epstein assumed that the coordinates $q_1,\ldots,q_n$ made the Hamilton-Jacobi equation completely separable. Then, for the action integral, the solution of the Hamilton-Jacobi equation becomes

$$W=\int\sum_{i=1}^{n} p_i dq_i=\sum_{i=1}^{n} W_i(q_i).$$

30. Because he considered real motions of the electron, $f(\zeta)\geq 0$ and $f(\eta)\geq 0$ were required. These conditions determine the two appropriate zero points for each function that have more zero points. See Epstein (1916b), pp. 497-501.

Epstein set Sommerfeld's quantum condition as

$$W_i = 2\int_{a_i}^{b_i} p_i dq_i = n_i h, \quad i = 1, \ldots, n, \tag{9}$$

and showed that the total energy depends on $(W_1, \ldots, W_n)$. Then, he explained degeneracy by considering the geometrical figures of the orbits for the degenerate system. He also noted that multiple coordinates make the Hamilton-Jacobi equations completely separable for the degenerate system.

Epstein introduced action-angle variables $(\omega_1, \ldots, \omega_n, \alpha_1, \ldots, \alpha_n)$ from Stäckel's general theory on the separable Hamilton-Jacobi equation. Epstein's procedure did not explicitly introduce the canonical transformation and was completely different from Schwarzschild's method. Noting that motions are conditionally periodic in the system that was described by the separable Hamilton-Jacobi equation, Epstein demonstrated a variable $\omega_i = n_i t + \beta_i$ with η_i a mean motion and β_i an initial value of the angle becomes canonical conjugate to W_i defined by Eq. 9 using Stäckel's formulas demonstrated in Charlier's books. Epstein next showed the action variables α_i were expressible as $\frac{1}{2\pi}W_i$, , *i.e.*, the action integral divide by 2π. He explicitly related action integrals to action variables while Schwarzschild did not so. In addition, the action variables are related to quantum numbers via the final identity:

$$\int_0^{2\pi} \frac{1}{2\pi} W_i d\omega_i = W_i = n_i h.$$

Then, the degeneracy can be discussed not only in terms of the action variables but also quantum numbers. Furthermore, Epstein established the degree of degeneracy: the number of combinations, Eq. 6, is called as the degree of degeneracy. In addition, Epstein argued that the orbits move on an $(n - s)$-dimensional surface when the degree of degeneracy is s.

3.3 Introduction of action-angle variables in Bohr's 1918 paper

Although Bohr's 1918 paper mentioned Schwarzschild's use of angle variables, it did not refer to Schwarzschild's introduction of action-angle variables though Schwarzschild actually used action variables as canonical conjugate of angle variables. Instead, Bohr wrote in the footnote "..., the connection between the notion of angle variables and the quantities *I*, discussed by Epstein in the latter paper, may be briefly exposed in following elegant manner...".[31] It is however essential for Bohr's discussion in the text of the 1918 paper that the action variables explicitly indicate the action integral. Bohr then insisted on Epstein's construction of the action-angle variables.

Combining Schwarzschild's and Epstein's ideas, Bohr introduced his definition of the action-angle variables in a footnote to his 1918 paper. Under the assumption that the variables of the Hamilton-Jacobi equation that correspond with Eq. 1 are completely separable, Bohr performed a canonical transformation on the equation of motion and derived action-angle variables for the dynamical system.[32] The former variables are the action integrals. Bohr wrote that this formulation had been suggested by Hendrik Kramers.

In the footnote, Bohr wrote that Jacobi's theorem was proven in Chapter 37 of Jacobi's *Vorlesungen*, which only has 36 chapters. The Jacobi theorem that Bohr mentioned had been called by Poincaré Jacobi's second theorem. As we mentioned before, this theorem had been proved not by Jacobi but by Charlier. Because Poincaré had associated Jacobi's name, Bohr as well as, mathematicians, physicists, and historians, seemed to have been confused and have found illusionary descriptions in Jacobi's *Vorlesungen*.

4. Conclusions

As we see now, the introduction and the development of the idea of action-angle variables were strongly connected to discoveries of mathematical properties of solvable Hamilton-Jacobi equations performed in the late nineteenth century and through the begin-

31. Bohr (1918), p.29-30. Here Epstein's latter paper means his second full paper. The quantities I denote $I_k = \int p_k(q_k, \alpha_1, \ldots, \alpha_n)dq_k \ \ (k = 1, \ldots, n)$, where $\alpha_1, \ldots, \alpha_n$ are arbitrary constants of a complete solution of the given Hamilton-Jacobi equation.

32. Bohr introduced expressions for the α_is in terms of the I_ks in the complete solution and used the modified complete solution as a generating function.

ning of the twentieth century. Almost all contributions that we have examined were written under the assumption that the variables of the Hamilton-Jacobi equation describing the motion of particles were completely separable. Stäckel showed that the solutions of a separable Hamilton-Jacobi equation were expressible as exact formulas. Using these formulas, he demonstrated that motions of the system that are associated with a separable Hamilton-Jacobi equation are conditionally periodic.

Based on Stäckel's result, mathematicians and physicists developed their ideas related to formulating the action-angle variables. Charlier proved that a complete solution of a separable Hamilton-Jacobi equation becomes a generating function of the canonical transformation. Noting Stäckel's formulas, Charlier attained new canonical equations that have an angle variable and an action integral as canonical coordinates. Schwarzschild generalized Charlier's idea and defined action-angle variables. He introduced the notion of degeneracy and showed that degeneracy occurred in the Stark effect. It is Epstein who explicitly connected action variables and action integral, and showed his action variables become canonical conjugate to angle variables referred to in Stäckel's formulas. The Bohr-Kramers definitions were introduced based on the definitions of Schwarzschild and Epstein.

Hence, a completely separable Hamilton-Jacobi equation is a sufficient condition for the occurrence of conditionally periodic motion and for defining action-angle variables in the system. The main interest for quantum physicists, including Bohr, was conditionally periodic motion. Because this motion also occurs in systems associated with non-solvable Hamilton-Jacobi equations, the assumption of a separable Hamilton-Jacobi equation was too strong for their purposes. Their next step was to drop this condition. This procedure is, however, beyond the scope of this paper.

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